FUNDAMENTAL SUBSPACES OF A MATRIX

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1. Transpose Transformations

Definition 1. Let A be an $m \times n$ matrix.

The *transpose* of A, which we denote by A^*, is the $n \times m$ matrix whose j^{th} row is the j^{th} column of A.

Definition 2. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

The *transpose* of T is the linear transformation $T^* : \mathbb{R}^m \to \mathbb{R}^n$ given by $T^*(w) = A^*w$, where $A = [T(e_1) \mid \cdots \mid T(e_n)]$.

Remark 1. Let A be an $m \times n$ matrix. Recall that A corresponds to a linear transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$ which is given by $T_A(v) = Av$. Then A^* corresponds to a linear transformation $T_{A^*} : \mathbb{R}^m \to \mathbb{R}^n$ which is given by $T_{A^*}(w) = A^*(w)$. Thus $T_A^* = T_{A^*}$.

Let $T = T_A$ be the transformation corresponding to A. We know that the columns of A are the destinations of the standard basis vectors of $\mathbb{R}^n$ under the transformation T. Thus the image of T is spanned by these vectors. On the other hand, the columns of T^* are the rows of A, so the image of T^* is a subspace of $\mathbb{R}^n$ which is spanned by the rows of A. We now investigate the relationship between the image of T^* and the kernel of T.

2. Column Spaces and Row Spaces

Definition 3. Let A be an $m \times n$ matrix.

The *image* of A is the image of T_A.

The *kernel* of A is the kernel of T_A.

The *column space* of A is the subspace of $\mathbb{R}^m$ spanned by the columns of A, and is denoted by $\operatorname{col}(A)$.

The *row space* of A is the subspace of $\mathbb{R}^n$ spanned by the rows of A, and is denoted by $\operatorname{row}(A)$.

The *null space* of A is the set $\{x \in \mathbb{R}^n \mid Ax = 0\}$, and is denoted by $\ker(A)$.

The *rank* of A is the dimension of the column space of A.

The *nullity* of A is the dimension of the null space of A.

Let A be an $m \times n$ matrix. The four fundamental subspaces associated to A are $\operatorname{col}(A)$, $\operatorname{row}(A)$, $\ker(A)$, and $\ker(A^*)$.

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Proposition 1. *Let A be an $m \times n$ matrix. Perform forward elimination on the matrix A to achieve $B = OA$, where O is invertible and B is in row echelon form. Perform backward elimination on B to achieve $C = UA$, where U is invertible and C is in reduced row echelon form. Then*

- **(a)** $\mathrm{col}(A) = \mathrm{row}(A^*)$;
- **(b)** $\mathrm{col}(A) = \mathrm{img}(T_A)$;
- **(c)** $\mathrm{row}(A) = \mathrm{img}(T_A^*)$;
- **(d)** *the rank of A is equal to the number of basic columns of B (or of C);*
- **(e)** *the nullity of A is equal to the number of free columns of B (or of C);*
- **(f)** $\ker(A) = \ker(B) = \ker(C)$;
- **(g)** $\mathrm{row}(A) = \mathrm{row}(B) = \mathrm{row}(C)$;
- **(h)** $\dim(\mathrm{col}(A)) = \dim(\mathrm{row}(A))$;
- **(i)** *the nonzero rows of B (or of C) form a basis for* $\mathrm{row}(A)$;
- **(j)** *the last $m - r$ rows of O (or of U) form a basis for* $\ker(A^*)$, *where r is the rank of A.*

Proof.

(a) $\mathrm{col}(A) = \mathrm{row}(A^*)$

This follows from the definition of transpose.

(b) $\mathrm{col}(A) = \mathrm{img}(T_A)$

This follows fact that the image of T_A is spanned by the destinations of the standard basis vectors; these destinations are the columns of A.

(c) $\mathrm{row}(A) = \mathrm{img}(T_A^*)$

This follows from **(a)** and **(b)**.

(d) *the rank of A is equal to the number of basic columns of B (or of C)*

The rank of B is clearly equal to the number of basic columns of B. The rank of A equals the rank of B because $B = UA$, where U is an invertible matrix. The transformation T_U is an isomorphism, so a basis for the image of T_A is sent by T_U to a basis for the image of T_B.

(e) *the nullity of A is equal to the number of free columns of B (or of C)*

The nullity of A is the number of free columns by the Rank Plus Nullity Theorem: $\dim(\ker(A)) = \dim(\ker(T_A)) = n - \dim(\mathrm{img}(T_A))$; since $\dim(\mathrm{img}(T_A))$ is the number of basic columns, $n - \dim(\mathrm{img}(T_A))$ must be the number of free columns.

(f) $\ker(A) = \ker(B) = \ker(C)$

This is given by the fact that composing on the left with an injective transformation does not change the kernel of a transformation. Since $B = OA$, we have $\ker(A) = \ker(T_A) = \ker(T_O \circ T_A) = \ker(T_{OA}) = \ker(OA) = \ker(B)$. Similarly, $\ker(A) = \ker(C)$.

(g) $\text{row}(A) = \text{row}(B) = \text{row}(C)$

If E is an elementary invertible matrix and D is any compatibly sized matrix, then the rows of ED are a linear combination of the rows of D; one sees this by considing the effect of the corresponding elementary row operation on D. Thus $\text{row}(ED) \subset \text{row}(D)$. But E^{-1} is also an elementary invertible matrix, so $\text{row}(D) = \text{row}(E^{-1}ED) \subset \text{row}(ED)$, which shows that $\text{row}(ED) = \text{row}(D)$ and E does not change the row space.

Since $B = OA$ and O is a product of elementary invertible matrices, we see that $\text{row}(B) = \text{row}(OA) = \text{row}(A)$. Similarly, $\text{row}(C) = \text{row}(A)$.

(h) $\dim(\text{col}(A)) = \dim(\text{row}(A))$

It is apparent from the definition of row echelon form that the nonzero rows of B form a basis for the row space of B.

By **(d)**, $\dim(\text{col}(A)) = \dim(\text{col}(B))$. The dimension of $\text{col}(B)$ is equal to the number of pivots in B (or C), which is equal to the number of nonzero rows of B (or C), which is equal to the dimension of $\text{row}(B)$. Thus $\dim(\text{col}(A)) = \dim(\text{col}(B)) = \dim(\text{row}(B)) = \dim(\text{row}(A))$.

(i) *the nonzero rows of B (or of C) form a basis for* $\text{row}(A)$

The nonzero rows of B (respectively C) form a basis for $\text{row}(B)$ (respectively $\text{row}(C)$). By **(g)**, $\text{row}(A) = \text{row}(B)$, and the result follows.

(j) *the last $m - r$ rows of O (or of U) form a basis for* $\ker(A^*)$

We show this for O; the proof for U is the identical.

Set $k = m - r$ and note that $\dim(\ker(A^*)) = k$. This follows from the Rank Plus Nullity Theorem and **(g)**: we have $r = \dim(\text{col}(A)) = \dim(\text{row}(A^*)) = \dim(\text{col}(A^*))$. Thus $\dim(\ker(A^*)) = m - \dim(\text{col}(A^*)) = m - r$.

Since O is invertible, its rows are linearly independent. Indeed, T_O is an isomorphism, so $\ker(O) = \{0\}$; thus $\dim(\text{row}(O)) = \dim(\text{col}(O)) = \dim(\mathbb{R}^m) - \dim(\ker(O)) = m$, since $\dim(\ker(O)) = 0$. Then $\text{row}(O) = \mathbb{R}^m$, so the rows of O are a basis for $\mathbb{R}^m$.

Thus the last k rows of O are linearly independent, so if these vectors are in $\ker(A^*)$, they are a basis for it. We only need to show that they are in $\ker(A^*)$.

Since $B = OA$, we have $B^* = A^*O^*$. The last k rows of B are zero, so the last k columns of B^* are zero. If x^* is one of the last k rows of O, then x is one of the last k columns of O^*, and A^*x is one of the last k columns of B^*; that is, it is zero. Thus x is in the kernel of A^*. □

3. Perpendicular Decompositions

Proposition 2. *Let $U \leq \mathbb{R}^n$. Set*

$$\perp(U) = \{v \in \mathbb{R}^n \mid u \cdot v = 0 \text{ for all } u \in U\}.$$

Then

(a) $\perp(U) \leq \mathbb{R}^n$;
(b) $U \cap \perp(U) = \{0\}$;
(c) $\perp(\perp(U)) = U$.

Proof. Exercise. □

Proposition 3. *Let A be an $m \times n$ matrix. Then*

(a) $\mathrm{row}(A) = \perp(\ker(A))$ *and* $\mathbb{R}^n = \mathrm{row}(A) \oplus \ker(A)$;
(b) $\mathrm{col}(A) = \perp(\ker(A^*))$ *and* $\mathbb{R}^m = \mathrm{col}(A) \oplus \ker(A^*)$.

Proof. In light of the fact that $\mathrm{col}(A) = \mathrm{row}(A^*)$, if we prove **(a)**, then **(b)** will follow simply by replacing A with A^*. Thus we prove **(a)**.

The coordinates of Ax are the dot products of the rows of A with the vector x. If $x \in \ker(A)$, the $Ax = 0$ (the zero vector). Thus each of the coordinates of Ax is equal to 0 (the zero scalar). This shows that each row of A is perpendicular to any vector in the kernel of A. Then any vector in the span of these rows is also perpendicular, because dot product is linear.

On the other hand, if x is not in the kernel, then it has a nonzero dot product with one of the rows, so it is not perpendicular to the row space. Therefore $\perp(\ker(A)) = \mathrm{row}(A)$.

Since the row space of A is perpendicular to the kernel of A, we see that $\mathrm{row}(A) \cap \ker(A) = \{0\}$. Now combine the Subspace Dimension Formula, the fact that $\dim(\mathrm{row}(A)) = \dim(\mathrm{col}(A))$, and the Rank plus Nullity Theorem to obtain

$$\begin{aligned}\dim(\mathrm{row}(A) + \ker(A)) &= \dim(\mathrm{row}(A)) + \dim(\ker(A)) + \dim(\mathrm{row}(A) \cap \ker(A)) \\ &= \dim(\mathrm{col}(A)) + \dim(\ker(A)) + 0 \\ &= \dim(\mathbb{R}^n).\end{aligned}$$

Since $\mathrm{row}(A) + \ker(A)$ is a subspace of $\mathbb{R}^n$ with the same dimension, it must be all of $\mathbb{R}^n$. Therefore $\mathbb{R}^n = \mathrm{row}(A) \oplus \ker(A)$. □

Corollary 1. *Let $U \leq \mathbb{R}^m$. Then $\mathbb{R}^m = U \oplus \perp(U)$.*

Proof. Let $\{u_1, \ldots, u_r\}$ be a basis for U. Form the matrix

$$A = [u_1 \mid \cdots \mid u_r \mid 0 \cdots \mid 0].$$

Then $\mathrm{col}(A) = U$, and $\perp(\mathrm{col}(A)) = \ker(A^*)$ with $\mathbb{R}^m = \mathrm{col}(A) \oplus \ker(A^*)$. □

Example 1. Let U be the subspace of $\mathbb{R}^m$ spanned by the vectors $\{v_1, \ldots, v_n\}$.

(a) Find a basis for U.
(b) Find a basis for $\perp(U)$.

Method of Solution. Form the $m \times n$ matrix $A = [v_1 \mid \cdots \mid v_n]$. Use forward elimination only to row reduce the augmented matrix $[A \mid I]$ to an augmented matrix $[B \mid O]$. A basis for U is given by the columns of A which correspond to the basic columns of B. Since $U = \mathrm{col}(A)$, a basis for $\perp(U)$ is given by the last $m - r$ rows of O, where $r = \dim(U)$. □

4. Exercises

Exercise 1. Let $U \leq \mathbb{R}^n$. Show that

(a) $\perp(U) \leq \mathbb{R}^n$;
(b) $U \cap \perp(U) = \{0\}$;
(c) $\perp(\perp(U)) = U$.

Exercise 2. Let

$$A = \begin{bmatrix} 2 & 0 & -1 & 4 & 1 \\ -2 & 0 & 2 & -2 & 0 \\ 0 & 0 & 1 & 2 & 2 \end{bmatrix}.$$

Let $T : \mathbb{R}^5 \to \mathbb{R}^3$ be the linear transformation given by $T(v) = Av$.

(a) Find a basis for $\operatorname{img}(T)$ and for $\ker(T)$.
(b) Find a basis for $\perp(\operatorname{img}(T))$ and for $\perp(\ker(T))$.

Exercise 3. Let U be the subspace of $\mathbb{R}^4$ spanned by the vectors

$$v_1 = (1, 0, -1, 1),\ v_2 = (2, 1, 1, 0),\ \text{and } v_3 = (0, -1, -3, 2).$$

(a) Find a basis for U.
(b) Find a basis for $\perp(U)$.
(c) Find a matrix A such that $U = \ker(A)$.

Exercise 4. Let

$$A = \begin{bmatrix} 1 & 2 & 0 & 2 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 2 & 1 & 1 & 0 & 1 \\ 0 & 4 & 0 & 5 & 1 \end{bmatrix}.$$

Find a basis for each of the four fundamental spaces associated to A.

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